# SYDNEY BOYS HIGH SCHOOL

# MATHEMATICS EXTENSION 2

## Trial Higher School Certificate 2001

Time Allowed: 3 hours (plus 5 minutes reading time)

Total Marks: 120

Examiner: Mr R Dowdell, Mr PS Parker

**INSTRUCTIONS:**

- Attempt *all* questions.
- *All* questions are of equal value.
- All necessary working should be shown in every question. Full marks may not be awarded if work is careless or badly arranged.
- Standard integrals are provided on the last page. Approved calculators may be used.
- Return your answers in 8 booklets, 1 for each question. Each booklet must show your name.
- If required, additional Writing Booklets may be obtained from the Examination Supervisor upon request.

NOTE: This is a trial paper only and does not necessarily reflect the content or format of the final Higher School Certificate Examination Paper for this subject.

**Question 1:** Marks

(a) Evaluate $\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9-x^2}}$ 2

(b) Find $\int x^3 e^{x^4+7}\, dx$ 2

(c) (i) Express $\frac{x^2+x+2}{(x^2+1)(x+1)}$ in the form $\frac{Ax+B}{x^2+1}+\frac{C}{x+1}$, where $A$, $B$ and $C$ are constants. 3

(ii) Hence find $\int \frac{x^2+x+2}{(x^2+1)(x+1)}\, dx$.

(d) Using integration by parts or otherwise, evaluate $\int_0^{\frac{1}{2}} \sin^{-1} x\, dx$ 3

(e) By using the substitution $x = \pi - y$, or otherwise, evaluate $\int_0^{\pi} x \sin^3 x\, dx$ 5

**Question 2: START A NEW BOOKLET** Marks

(a) $\dfrac{4+3i}{1+\sqrt{2}\,i} = a + ib$, for $a$, $b$ real. 2

Find the exact values of $a$ and $b$.

(b) Given $z = 1 - \sqrt{3}\,i$, 3

(i) show that $z^2$ is a real multiple of $\dfrac{1}{z}$;

(ii) plot $z$, $z^2$, $\dfrac{1}{z}$ on an Argand diagram.

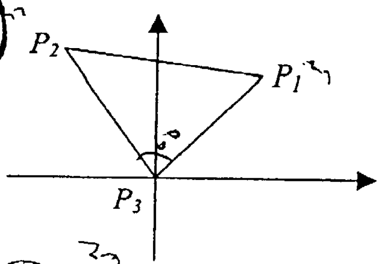
(c) Sketch the region represented by 2

$|z| \le 4$ and $\dfrac{\pi}{3} < \arg z \le \dfrac{2\pi}{3}$.

(d) (i) Show that $\dfrac{(1+i\sqrt{3})^6}{(\sqrt{3}-i)^k} = 2^{6-k}\,cis\left(\dfrac{k\pi}{6}\right)$. 4

(ii) For what values of $k$ is $\dfrac{(1+i\sqrt{3})^6}{(\sqrt{3}-i)^k}$ purely imaginary?

(e)



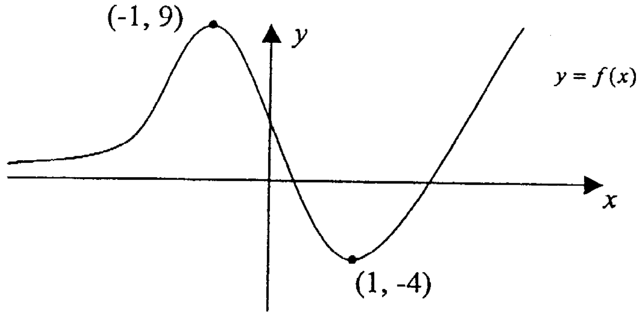

The points $P_1$, $P_2$ and $P_3$ represent the complex numbers $z_1$, $z_2$ and $z_3$ respectively. (NOTE: $z_3 = 0$.) 4

(i) If $P_1$, $P_2$ and $P_3$ are the vertices of an equilateral triangle, show that $\dfrac{z_2}{z_1} = \dfrac{1+i\sqrt{3}}{2}$ and deduce that $z_1^{\,2} + z_2^{\,2} = z_1 z_2$.

(ii) Deduce that if $z_1, z_2$ and $z_3$ are **ANY** three complex numbers at the vertices of an equilateral triangle then $z_1^{\,2} + z_2^{\,2} + z_3^{\,2} = z_1 z_2 + z_2 z_3 + z_3 z_1$

**Question 3: START A NEW BOOKLET** Marks

(a) If the curve below represents $y = f(x)$, 12




make neat sketches, on separate axes, of

(i) $y = (f(x))^2$

(ii) $y = \dfrac{1}{f(x)}$

(iii) $y = |f(x)|$

(iv) $y = f(|x|)$

(v) $y^2 = f(x)$

(vi) $y = f'(x)$

(b) Two sides of a triangle are in the ratio 3:1 and the angles opposite these sides differ by $\dfrac{\pi}{6}$. Show that the smaller of the two angles is $\tan^{-1}\left(\dfrac{1}{6-\sqrt{3}}\right)$. 3

**Question 4:** **START A NEW BOOKLET** **Marks**

(a) $1+i$ and $3-i$ are zeroes of a real, monic polynomial, $p(x)$, of degree 4. **3**

(i) Express $p(x)$ as a product of two real quadratic factors.

(ii) Explain briefly why the polynomial $p(x)$ cannot take negative values.

(b) $x^3+3px+q=0$ has a double root of $x=k$. **4**

(i) Show that $p=-k^2$.

(ii) Show that $4p^3+q^2=0$.

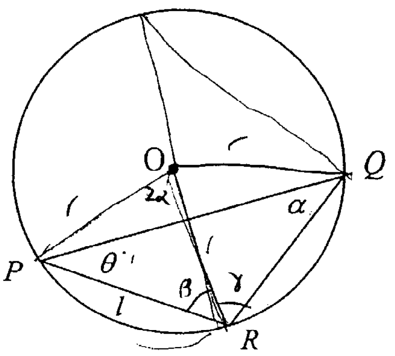
(iii) Hence factorise $x^3-6ix+4-4i$ into linear factors, given that it has a repeated factor.

(c) Consider $f(x)=x^3+9x+26$ and $g(x)=x^2+26x-27$. **3**

(i) Verify that $f\left(x-\dfrac{3}{x}\right)=\dfrac{g(x^3)}{x^3}$.

(ii) Hence solve $f(x)=0$.

(d) **5**




$\Delta PQR$ is a triangle inscribed in a circle of radius $r$. $PR$ has length $l$, and $\angle PQR=\alpha$

(i) Show that $l=2r\sin\alpha$.

(ii) If $\angle QPR=\theta$, show that the area of $\Delta PQR$ is $r^2\sin\alpha(\cos\alpha-\cos(2\theta+\alpha))$

(iii) If $PQ=QR$, what is the area of $\Delta PQR$ in terms of $r$ and $\alpha$ ?

## Question 5: START A NEW BOOKLET

Marks

(a) A mass of $m$ kilograms falls from rest. It experiences resistance during its fall equal to $mkv$ where $v$ is its speed in metres per second and $k$ is a positive constant. Let $x$ be the distance in metres of the mass from its starting point measured positively as it falls and $t$ be the time in seconds. 8

(i) Show that the equation of motion of the mass is $\ddot{x} = g - kv$ where $g$ is the acceleration due to gravity.

(ii) Show that the terminal velocity is $\frac{g}{k}$.

(iii) Find $v$ as a function of $t$.

(iv) Find $x$ as a function of $t$.

(b) (i) In how many ways can 10 students be grouped into two teams of 5 to play a game of basketball? 2

(ii) Two of the 10 students are twins. If the teams are formed at random, what is the probability that the twins play on the same team?

(c) A group of men and women is seated randomly around a circular table. What is the probability that none of the men are sitting next to each other if there are 5

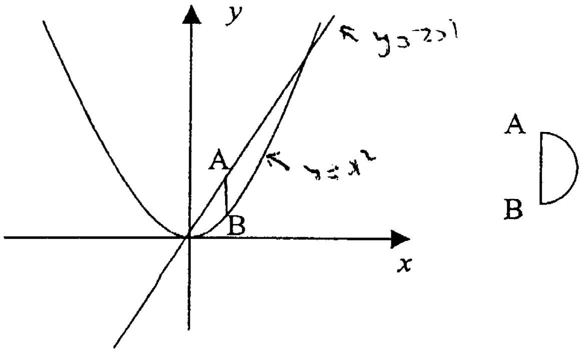
(i) 3 men and 2 women;

(ii) 2 men and 3 women;

(iii) $n$ men and $n + 1$ women?

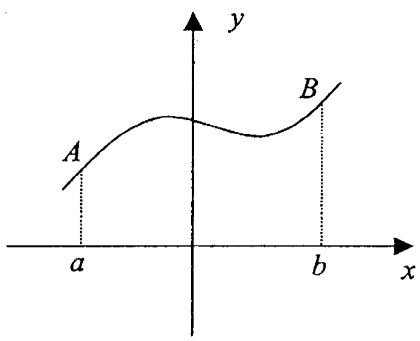
**Question 6: START A NEW BOOKLET** Marks

(a) The base of a solid is the region enclosed by $y = 2x$ and $y = x^2$. Cross sections taken perpendicular to the $x$ axis are semicircles with the diameter in the base of the solid (as indicated the diameter AB of the semicircle is perpendicular to the $x$ axis; the semicircle is perpendicular to the $xy$ plane). 5



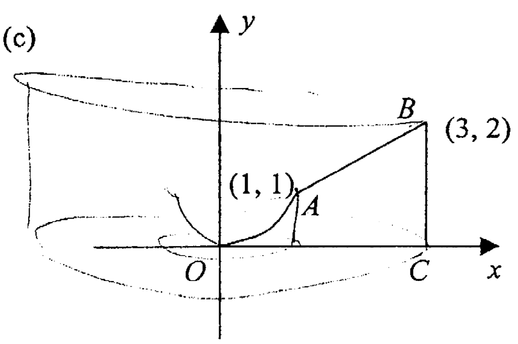

Find the volume of the solid.

(b) The length of the arc $AB$ on the curve $y = f(x)$ between $x = a$ and $x = b$ is given by $l = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$. 4




Find the length of the arc on $y = x^{\frac{3}{2}}$ between $x = 0$ and $x = 4$.

(c)

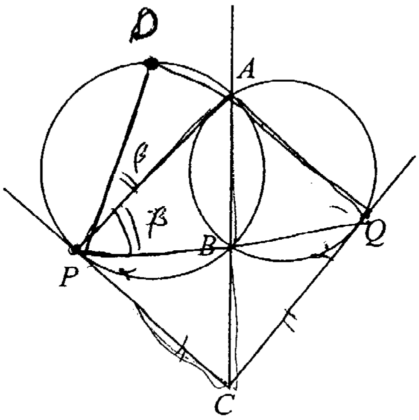



$OA$ is an arc of the parabola $y = x^2$. The region $OABC$ is rotated about the $y$ axis forming a bowl. By using cylindrical shells determine the holding capacity of the bowl. 6

**Question 7: START A NEW BOOKLET** — Marks

(a) Find the value of $a$ given that $\left(\sqrt{x}+\dfrac{a}{x}\right)^{10}$ has 13440 as coefficient of $x^{-4}$. 3

(b) 5




Two circles intersect at $A$ and $B$. $AB$ is produced to a point $C$, such that $CP$ and $CQ$ are tangents to the circles as shown and $PBQ$ is a straight line.

NOTE: The diagram is not drawn to scale.

(i) Express $CP$ in terms of $CB$ and $CA$, and hence prove that $CP = CQ$.

(ii) Show that $A$, $P$, $C$ and $Q$ are concyclic.

(iii) Let $QA$ produced meet the larger circle at D. Show that $PB$ bisects $\angle CPD$.

(c) Let $T(m,y)=\dfrac{{}^{m}C_0}{y}-\dfrac{{}^{m}C_1}{y+1}+\dfrac{{}^{m}C_2}{y+2}-\ \ldots\ldots\ +(-1)^m\dfrac{{}^{m}C_m}{y+m}$. 7

(i) If it is given that $T(k,x)=\dfrac{k!}{x(x+1)(x+2)\ldots(x+k)}$ for a particular value of $k$, show that

$$T(k,x)-T(k,x+1)=T(k+1,x)$$

(ii) Hence prove, using Mathematical Induction or otherwise, that for $n \geq 1$

$$T(n,x)=\frac{{}^{n}C_0}{x}-\frac{{}^{n}C_1}{x+1}+\frac{{}^{n}C_2}{x+2}-\ \ldots\ +(-1)^n\frac{{}^{n}C_n}{x+n}=\frac{n!}{x(x+1)(x+2)\ldots(x+n)}$$

(NOTE: you may use without proof the result ${}^{m+1}C_r={}^{m}C_r+{}^{m}C_{r-1}$)

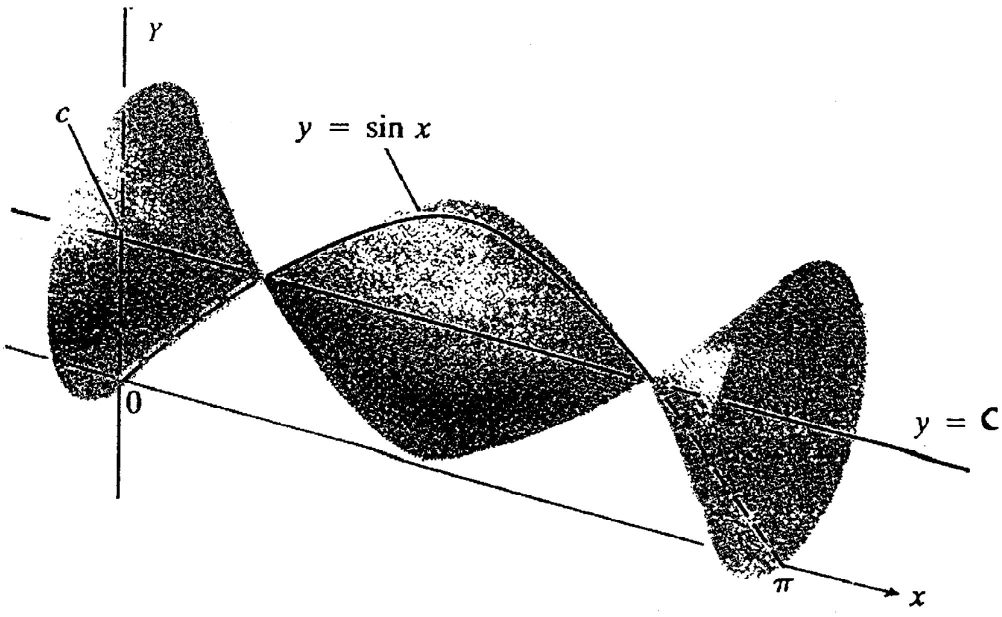
(iii) Hence prove that

$$\frac{{}^{n}C_0}{1}-\frac{{}^{n}C_1}{3}+\frac{{}^{n}C_2}{5}-\ \ldots\ +(-1)^n\frac{{}^{n}C_n}{2n+1}=\frac{2^n n!}{1.3.5\ldots(2n+1)}$$

## Question 8: START A NEW BOOKLET

Marks

(a) 5




The arch $y = \sin x$, $0 \le x \le \pi$ is revolved around the line $y = c$ to generate the solid shown. Find the value of $c$ that minimises the volume.

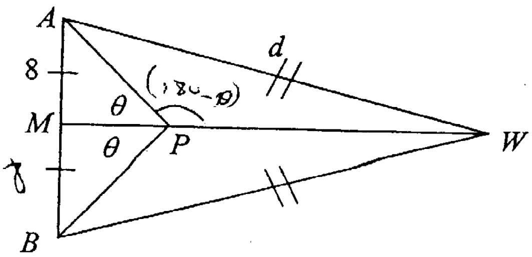
**Question 8 is continued on Page 10**

(b) (i) Let $f(\theta) = \dfrac{2 - \cos\theta}{\sin\theta}$, $0 < \theta < \dfrac{\pi}{2}$.

Show that $f'(\theta) = \dfrac{1 - 2\cos\theta}{\sin^2\theta}$.

Find the minimum value of $f(\theta)$.

(ii) Two towns $A$ and $B$ are 16km apart, and each at a distance of $d$ km from a water well at $W$. Let $M$ be the midpoint of $AB$, $P$ be a point on the line segment $MW$, and $\theta = \angle APM = \angle BPM$. The two towns are to be supplied with water from $W$, via three straight water pipes: $PW$, $PA$ and $PB$ as shown below.




Show that the total length of the water pipe $L$ is given by $L = 8f(\theta) + \sqrt{d^2 - 64}$, when $\dfrac{8}{d} \le \sin\theta \le 1$, where $f(\theta)$ is given in part (i).

(iii) If $d = 20$, find the length of $MP$ when $L$ is minimum, and the minimum value of $L$.

Show that this minimum value of $L$ is less than the sum of any pair of sides of $\Delta ABW$.

(iv) If $d = 9$, show that the minimum value of $L$ cannot be found by using the same methods as used in part (iii). Explain briefly how to find the minimum value of $L$ in this case. (Hint: Draw a diagram which illustrates this situation.)

**END OF PAPER**

SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

**2001**

**TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION**

# Mathematics Extension 2

# Sample Solutions

QUESTION 1.

(a) $\displaystyle\int_0^{3/2} \frac{dx}{\sqrt{9-x^2}} = \left[\sin^{-1}\frac{x}{3}\right]_0^{3/2}$

$= \sin^{-1}\frac{1}{2} - \sin^{-1} 0.$

$= \frac{\pi}{6}.$ ✓ ✓ (2)

(b) $\displaystyle\int x^3 e^{x^4+7}\,dx = \frac{1}{4} e^{x^4+7} + c$ ✓ ✓ (2)

(c) (i). $\displaystyle\frac{x^2+x+2}{(x^2+1)(x+1)} \equiv \frac{Ax+B}{x^2+1} + \frac{C}{x+1} \Rightarrow x^2+x+2 = (Ax+B)(x+1) + C(x^2+1)$

$= Ax^2+Ax+Bx+B+Cx^2+C$

$= (A+C)x^2 + (A+B)x + B+C.$

$\therefore \displaystyle\int \frac{x^2+x+2}{(x^2+1)(x+1)}\,dx = \int\left(\frac{1}{x^2+1} + \frac{1}{x+1}\right)dx$ :

$= \tan^{-1}x + \ln(x+1) + c$ (1)

$A+C = 1$
$A+B = 1$ } $B - C = 0$ — (1)
$B+C = 2$ — (2)
(1) + (2)
$2B = 2$
$B = 1$
$\therefore C = 1$
$A = 0.$ (1) (3)

(d) $\displaystyle\int_0^{1/2} \sin^{-1}x\,dx = \int_0^{1/2} \frac{d}{dx}(x).\sin^{-1}x\,dx$

$= \left[x\sin^{-1}x\right]_0^{1/2} - \displaystyle\int_0^{1/2} x.\frac{1}{\sqrt{1-x^2}}\,dx$ ✓

$= \frac{1}{2}\times\frac{\pi}{6} + \left[\sqrt{1-x^2}\right]_0^{1/2}$ ✓

$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$ (3)

(e) $I = \displaystyle\int_0^{\pi} x.\sin^3 x\,dx.$

$= -\displaystyle\int_{\pi}^{0} (\pi-y).\sin^3(\pi-y)\,dy.$

$= \displaystyle\int_0^{\pi} (\pi-y)\sin^3 y\,dy.$

$= \displaystyle\int_0^{\pi} (\pi-x)\sin^3 x\,dx.$ 1

$I = \displaystyle\int_0^{\pi} \pi\sin^3 x\,dx - \int_0^{\pi} x\sin^3 x\,dx.$ 2 (5)

$\therefore 2I = \pi\displaystyle\int_0^{\pi} \sin^2 x.\sin x\,dx$ let $u = \cos x.$ 3 $du = -\sin x\,dx$

$= \pi\displaystyle\int_1^{-1} (1-u^2).-du.$ 4

$= \pi\displaystyle\int_{-1}^{1} (1-u^2)\,du = 2\pi\int_0^1 (1-u^2)\,du = 2\pi\left[u - \frac{1}{3}u^3\right]_0^1$ 5

$\therefore I = \pi\left[1-\frac{1}{3}\right] = \frac{2\pi}{3}.$

QUESTION 2 (a) $\frac{4+3i}{1+\sqrt{2}i} = \frac{4+3i}{1+\sqrt{2}i} \times \frac{1-\sqrt{2}i}{1-\sqrt{2}i}$

$= \frac{4-4\sqrt{2}i+3i+3\sqrt{2}}{3}$

$= \frac{4+3\sqrt{2}+i(3-4\sqrt{2})}{3}$

$\therefore \boxed{a = \frac{4+3\sqrt{2}}{3},\ b = \frac{3-4\sqrt{2}}{3}}$ ✓ ✓ 2

(b) $z = 1-\sqrt{3}i$

$z^2 = \cancel{-2}-2\sqrt{3}i$

$= -2(1+\sqrt{3}i)$

$\frac{1}{z} = \frac{1}{1-\sqrt{3}i} \times \frac{1+\sqrt{3}i}{1+\sqrt{3}i}$

$= \frac{1+\sqrt{3}i}{4}$

$= \frac{1}{4}(1+\sqrt{3}i)$

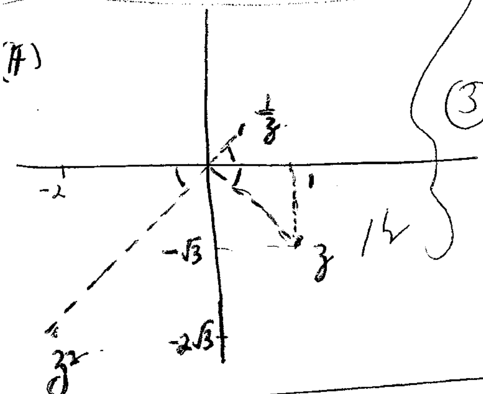
$z = 2\,\text{cis}\,-\frac{\pi}{3}$

$z^2 = 4\,\text{cis}\,-\frac{2\pi}{3}$

$\frac{1}{z} = \frac{1}{2}\,\text{cis}\,\frac{\pi}{3}$

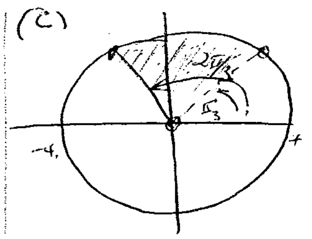
$\therefore z^2 = -8 \times \frac{1}{z}$ $1\frac{1}{2}$ → OR SHOW $z^3 = -8$.

(ii)




$1\frac{1}{2}$ 3

(c)




2

(d)(i) $\frac{(1+i\sqrt{3})^6}{(\sqrt{3}-i)^k} = \frac{\left(2\,\text{cis}\,\frac{\pi}{3}\right)^6}{\left(2\,\text{cis}\left(-\frac{\pi}{6}\right)\right)^k}$

$= 2^{6-k}\,\text{cis}\left(2\pi + \frac{k\pi}{6}\right)$

$= 2^{6-k}\,\text{cis}\,\frac{k\pi}{6}$ 2

(ii) Pure imag. $\cos\frac{k\pi}{6} = 0$

$\frac{k\pi}{6} = 2n\pi \pm \frac{\pi}{2}$

$\frac{k}{6} = \cancel{2}\,2n \pm \frac{1}{2}$

$\boxed{k = 3(4n \pm 1)}$ $n \in J$

$k = \pm 3$, etc — -

$12n \pm 3 \Rightarrow \cancel{\pm 3, \pm 9, \pm 21}$ - -

$3, \cancel{15}, 9, 15, 21, -$ -

$\cancel{-3, -9, -15}$

2

$\pm 3, \pm 9, \pm 15, \pm 21, -$ -

OR $(6n+3)$ $n \in J$.

(e). (i) now $z_2 = \text{cis}\frac{\pi}{3} \cdot z_1$

$\therefore \frac{z_2}{z_1} = \text{cis}\frac{\pi}{3} = \frac{1+i\sqrt{3}}{2}$ —— 1

$\therefore z_1^2 + z_2^2 = z_1^2 + (\text{cis}\frac{\pi}{3} z_1)^2$

$= z_1^2 (1 + \text{cis}\frac{2\pi}{3})$

$= z_1^2 (1 + -\frac{1}{2} + \frac{i\sqrt{3}}{2})$

$= z_1^2 (+\frac{1}{2} + i\frac{\sqrt{3}}{2})$

$= z_1^2 \times \frac{z_2}{z_1}$

$= z_1 z_2$. —— 1

(ii) clearly $(z_1 - z_3)^2 + (z_2 - z_3)^2 = (z_1 - z_3)(z_2 - z_3)$

$z_1^2 - 2z_1z_3 + z_3^2 + z_2^2 - 2z_2z_3 + z_3^2 = \cancel{z_2 - z_3 z_3}$

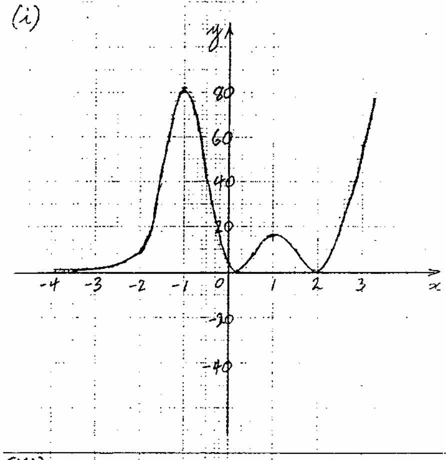
$= z_1z_2 - z_1z_3 - z_3z_2 + z_3^2$.

$\therefore z_1^2 + z_2^2 + z_3^2 = z_1z_3 + z_2z_3 + z_1z_2$. Q.E.D.

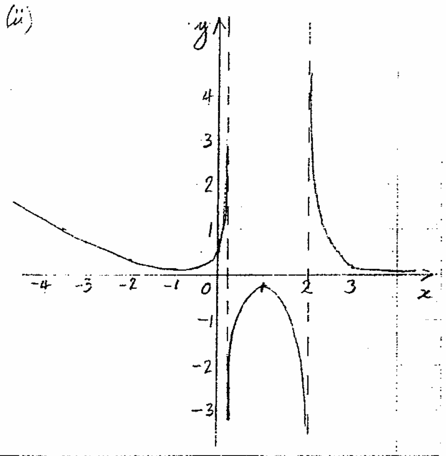
2

3(a)

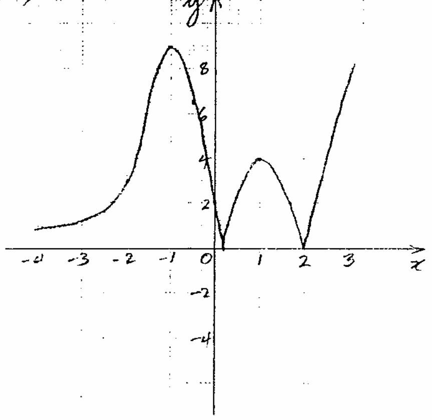
(i)




(ii)




(iii)

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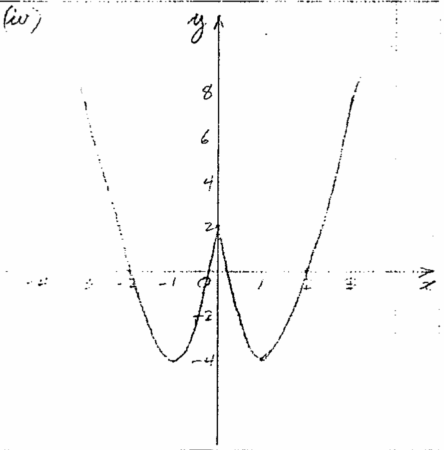
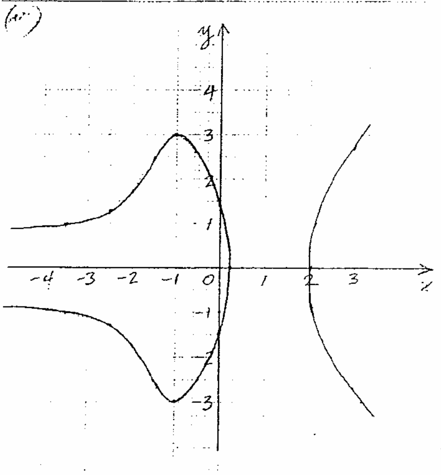
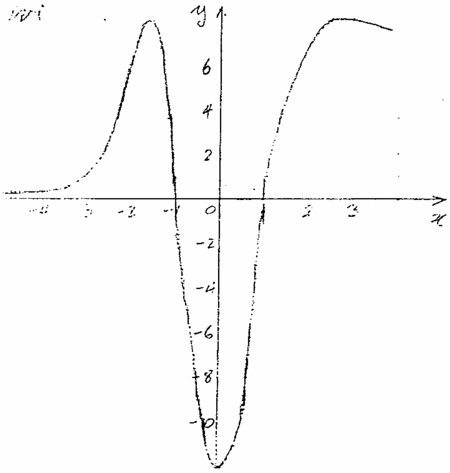


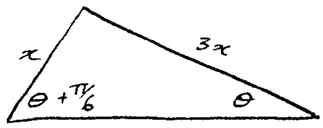

(v)




(vi)



3(b)




$$\frac{\sin\theta}{x} = \frac{\sin(\theta+\pi/6)}{3x}$$

$$3\sin\theta = \sin\theta\cos\pi/6 + \cos\theta\sin$$

$$3\sin\theta = \frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta$$

$$6\sin\theta - \sqrt{3}\sin\theta = \cos\theta$$

$$\frac{\sin\theta}{\cos\theta}\left\{6-\sqrt{3}\right\} = 1$$

$$\tan\theta = \frac{1}{6-\sqrt{3}}$$

$$\therefore \theta = \tan^{-1}\left(\frac{1}{6-\sqrt{3}}\right).$$

4(a) (i) zeroes $1 \pm i$, $3 \pm i$

$$[x-(1+i)][x-(1-i)] = x^2 - x + ix - x - ix + 1 + 1$$

$$= x^2 - 2x + 2$$

$$[x-(3+i)][x-(3-i)] = x^2 - 6x + 10$$

$$\therefore P(x) = (x^2-2x+2)(x^2-6x+10)$$

(ii) Both factors are positive definite ($\Delta < 0$, coefft. of $x^2 > 0$) so their product must be positive.

4(b) (i) $P(x) = x^3 + 3px + q$

$$P'(x) = 3x^2 + 3p$$

$$P'(k) = 3k^2 + 3p = 0$$

$$\therefore p = -k^2.$$

(ii) $P(k) = k^3 + 3pk + q = 0$

$$k = (-p)^{1/2}$$

$$\therefore -p\cdot(-p)^{1/2} + 3p(-p)^{1/2} + q = 0$$

$$\frac{q}{2p} = (-p)^{1/2}$$

$$\frac{q^2}{4p^2} = -p$$

$$4p^3 + q^2 = 0$$

4(b) (iii) $x^3 + 3(-2i)x + (4-4i) = P(x)$

Let zeroes be $k, k, \omega$

$-k^2 = -2i$

$k = \sqrt{2i}$

$k = \pm(1+i)$ ✓

$k^2\omega = -4+4i$

$\omega = \frac{-4+4i}{2i} \times \frac{2i}{2i}$

$= \frac{+8+8i}{+4}$

$= 2+2i$

$2k + \omega = \cancel{4i}\ 0$

$2k = -2-2i$

$k = -1-i$ ✓

So $P(x) = (x+1+i)^2(x-2-2i)$.

$(a+ib)^2 = 2i$

$a^2 - b^2 + 2aib = 2i$

$a^2 - b^2 = 0$

$a^2 + b^2 = 2$

$2a^2 = 2$

$a = \pm b = \pm \cancel{2}$

but $ab = \cancel{2}\ 1$

$\therefore a = b = \pm 1$

4(c) (i) LHS $= \left(x - \frac{3}{x}\right)^3 + 9\left(x - \frac{3}{x}\right) + 26$

$= x^3 - 3.\cancel{x^2}.\frac{3}{x} + 3.x.\cancel{\frac{9}{x^2}} - \frac{27}{x^3} + \cancel{9x} - \cancel{\frac{27}{x}} + 26$

$= x^3 + 26 - {}^{27}/_{x^3}$

RHS $= \frac{x^6 + 26x^3 - 27}{x^3}$ ✓

$=$ LHS.

(ii) $\frac{g(x^3)}{x^3} = \frac{(x^3+27)(x^3-1)}{x^3}$

Put $x = p - \frac{3}{p}$

$f(x) = 0 = \frac{(p^3+27)(p^3-1)}{p^3}$

$\therefore p = -3, 1$ if $p \neq 0$

$x = -2$ ✓

$f(x) = 0 = (x+2)(x^2 - 2x + 13)$

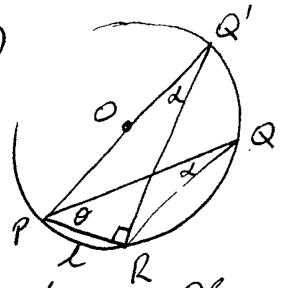
$\therefore x = -2,\ 1 \pm 2\sqrt{3}\,i$ ✓

$$\begin{array}{r|l} & x^2 - 2x + 13 \\ \hline x+2 & x^3 + 0x^2 + 9x + 26 \\ & x^3 + 2x^2 \\ \hline & -2x^2 + 9x \\ & -2x^2 - 4x \\ \hline & 13x + 26 \end{array}$$

$x = \frac{2 \pm \sqrt{4-52}}{2}$

$= 1 \pm 2\sqrt{3}\,i$

4(d)(i)




$P\hat{Q}'R = \alpha$ (angle standing on same arc

$P\hat{R}Q' = \pi/2$ (angle in semicircle)

$PQ' = 2r$ (diameter

$\sin\alpha = \dfrac{l}{2r}$

$l = 2r\sin\alpha.$ ✓

(ii) $\dfrac{l}{\sin\alpha} = \dfrac{QR}{\sin\theta}$ $\qquad QR = \dfrac{l\sin\theta}{\sin\alpha}$

$P\hat{R}Q = \pi - \theta - \alpha$

$c_2 = 2c^2 - 1$

$c^2 = \dfrac{1+c_2}{2}$

$\text{Area} = \frac{1}{2}.PR.RQ\sin P\hat{R}Q$

$= \dfrac{l}{2}.\dfrac{l\sin\theta}{\sin\alpha}.\sin(\pi-\theta-\alpha)$ ✓

$= \dfrac{{}^2\cancel{4}r^2\sin^2\alpha\sin\theta\sin(\theta+\alpha)}{\cancel{2}\cancel{\sin\alpha}}$ ✓

$= 2r^2\sin\alpha\{\sin\theta(\sin\theta\cos\alpha + \cos\theta\sin\alpha)\}$

$= 2r^2\sin\alpha\{(1-\cos^2\theta)\cos\alpha + \sin\theta\cos\theta\sin\alpha\}$

$= r^2\sin\alpha\{(2-1-\cos2\theta)\cos\alpha + \sin2\theta\sin\alpha\}$

$= r^2\sin\alpha\{\cos\alpha - \cos(2\theta+\alpha)\}$

(iii) If $PQ = QR$ then $2\theta + \alpha = \pi$ (isosceles $\Delta$)

$\text{Area} = r^2\sin\alpha(\cos\alpha - \cos\pi)$

$= r^2\sin\alpha(\cos\alpha + 1)$ ✓

Question 5

(a) (i) Taking the downward direction as positive we get the following force equation $m\ddot{x} = mg - mkv$, the resistance is negative as it OPPOSES the motion and is therefore directed upwards.

So $m\ddot{x} = mg - mkv \Rightarrow \ddot{x} = g - kv$

(ii) The terminal velocity is when the net acceleration of the mass is 0 ie $\ddot{x} = g - kv = 0 \Rightarrow V_T = g/k$.

(iii) Taking $\ddot{x} = \frac{dv}{dt} \Rightarrow \frac{dv}{dt} = g - kv$

$$\frac{dt}{dv} = \frac{1}{g-kv} = -\frac{1}{k}\left(\frac{-k}{g-kv}\right)$$

$$t = -\frac{1}{k}\ln|g-kv| + c$$

$$t = 0, v = 0 \Rightarrow c = \frac{1}{k}\ln g$$

$$\therefore t = -\frac{1}{k}\ln\left|\frac{g-kv}{g}\right| \Rightarrow e^{-tk} = \left|\frac{g-kv}{g}\right| = \frac{g-kv}{g}$$

We can remove the absolute value brackets since the initial direction is positive and it doesn't come to rest until it hits the ground so $\frac{g-kv}{g} > 0$

$$g - kv = ge^{-tk} \Rightarrow v = \frac{g}{k}(1 - e^{-tk})$$

(iv) $v = \frac{g}{k}(1 - e^{-tk}) \Rightarrow x = \int \frac{g}{k}(1 - e^{-tk})dt$

$$x = \frac{g}{k}\left(t + \frac{1}{k}e^{-tk}\right) + c_1$$

$$t = 0, x = 0 \Rightarrow c_1 = -\frac{g}{k^2}$$

$$\therefore x = \frac{g}{k}\left(t + \frac{1}{k}e^{-tk} - \frac{1}{k}\right)$$

(b) (i) If we choose 5 players to form a team, this can be done in $\binom{10}{5}$ ways. But the remaining 5 will form the opposing team. So there are $\frac{1}{2}\binom{10}{5} = 126$

(ii) If the twins are on one team then the remaining team can be formed in $\binom{8}{3}$ = 56 ways. So the probability that the twins are on the same team is $56/126 = 4/9$.

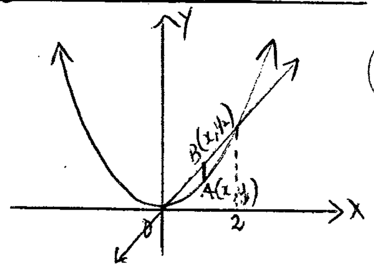
(5) (c) (i) There are 4! = 24 ways to arrange everyone without restrictions. However with 3 men and only 2 women there must be one pair of men sitting next to each other.
Probability = 0.

(ii) There are 4! ways to sit everyone down without restriction.
Seat two women down next to each other. This leaves 2! = 2 ways to seat the men down and 1 way to sit the other woman.
The two women can be chosen in $\binom{3}{2} = 3$ ways. Then the two women seated together can swap seats
So there are $2 \times 3 \times 2 = 12$ ways. to sit everyone down.
So the probability is $12/24 = 1/2$.

(iii) There are $(2n!)$ ways to sit everyone down without restrictions.
Choose two women first and sit them down. This can be done in $\binom{n+1}{2} = \frac{n(n+1)}{2}$ ways. The remaining women can be seated in $(n-1)!$ ways. Then the men can be seated in $n!$ ways.
The two women together can be swapped around.
A total of $\frac{n(n+1)}{2} \times (n-1)! \times n! \times 2 = (n+1)(n!)^2$.
So the probability is $\frac{(n+1)(n!)^2}{(2n!)} = \frac{(n+1)!n!}{(2n!)}$.

## QUESTION 6

i)




5

$x^2=2x \Rightarrow x=0$ or $2$

Diameter $= y_2-y_1 = 2x-x^2$

$A$ = Area of semi-circle $= \frac{\pi}{2}\left(x-\frac{x^2}{2}\right)^2$ 1

Thickness of solid $\delta x$.

Vol. of element $= \delta V = A\,\delta x$

$= \frac{\pi}{2}\left(x-\frac{x^2}{2}\right)^2\delta x$ 1

Total Volume $= \lim_{\delta x\to 0}\sum_{x=0}^{2}\frac{\pi}{2}\left(x-\frac{x^2}{2}\right)^2\delta x$

$= \frac{\pi}{2}\int_0^2\left(x-\frac{x^2}{2}\right)^2dx$

$= \frac{\pi}{2}\int_0^2\left(x^2-x^3+\frac{x^4}{4}\right)dx$ 1

$= \frac{\pi}{2}\left[\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{20}\right]_0^2$ 1

$= \frac{\pi}{2}\left[\frac{8}{3}-4+\frac{32}{20}-0\right]$

$= \frac{2\pi}{15}$ units³. 1

ii) $l=\int_0^4\sqrt{1+\left(\frac{3}{2}\sqrt{x}\right)^2}\,dx$

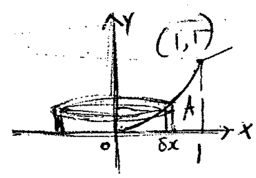
$=\int_0^4\sqrt{1+\frac{9}{4}x}\,dx$ 1 4

$=\left[\frac{\left(\frac{9}{4}x+1\right)^{3/2}}{\frac{3}{2}\cdot\frac{9}{4}}\right]_0^4$

$=\frac{8}{27}\left[10^{3/2}-1^{3/2}\right]$ 1

$=\frac{8}{27}\left[10\sqrt{10}-1\right]$ units.

(c)




Height = y.

Vol. when region A is rotated about Y axis

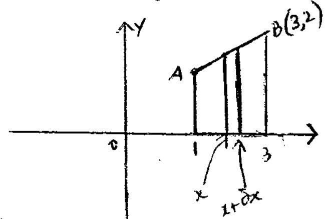
Vol. of shell $\delta V=\pi\left[(x+\delta x)^2-x^2\right]y$

$=\pi\left[x^2+2x\delta x+\delta x^2-x^2\right]y$

$=2\pi xy\,\delta x$

$V=2\pi\int_0^1 x\times x^2\,dx$ 1

3 $=2\pi\left[\frac{x^4}{4}\right]_0^1=\frac{\pi}{2}$ units³ 1




Eqⁿ of line AB is $x=2y-1$ ½

or $y=\frac{x+1}{2}$

$V=2\pi\int_1^3 x\left(\frac{x+1}{2}\right)dx$ 1

$=\pi\int_1^3(x^2+x)\,dx$

3 $=\pi\left[\frac{x^3}{3}+\frac{x^2}{2}\right]_1^3$ 1

$=\pi\left[\left(9+\frac{9}{2}\right)-\left(\frac{1}{3}+\frac{1}{2}\right)\right]=\frac{38\pi}{3}$

½

∴ Total volume $=\frac{\pi}{2}+\frac{38\pi}{3}=\frac{79\pi}{6}$

Vol. of bowl $=18\pi-\frac{79\pi}{6}=\frac{29\pi}{6}$ units³

Question 7

(a) $$\left(\sqrt{x}+\frac{a}{x}\right)^{10}=\left(x^{\frac{1}{2}}+ax^{-1}\right)^{10}=\sum_{r=0}^{10}{}^{n}C_r\left(x^{\frac{1}{2}}\right)^{10-r}\left(ax^{-1}\right)^{r}=\sum_{r=0}^{10}\left(a^r\times{}^{n}C_r\right)x^{5-3r/2}$$

To get the coefficient of $x^{-4}$ we need $5-3r/2=-4\Rightarrow -3r/2=-9\Rightarrow r=6$
So the coefficient of $x^{-4}=a^6\times^{10}C_6=13440$
$$a^6=\frac{13440}{^{10}C_6}=64\Rightarrow a=2$$

(b) (i) $CP^2=AC\times BC$ by the rule for intercepts and tangents.
Similarly $CQ^2=AC\times BC\Rightarrow CP^2=CQ^2\Rightarrow CP=CQ$ as ($CP$, $CQ>0$)

(ii) $\Delta PQC$ is isosceles with $\angle CPQ=\angle CQP=x$ (base angles of isos. $\Delta$)
$\angle BAP=\angle CPQ=x$ (alternate segment theorem). Similarly $\angle BAQ=x$
So $\angle CAQ=\angle CPQ$ both of which stand on chord $CQ$.
So $A$, $P$, $C$ & $Q$ are concyclic points by the converse of angles in the same segment theorem.

(iii) The exterior angle $\angle DAP=180-2x$, $DAQ$ is a straight line.
So $\angle DAC=180-x$. Now $BADP$ is a cyclic quad so that
$\angle BPD+\angle BAD=180°\ \Rightarrow\angle BAD=x$.
Thus $\angle CPB=\angle BPD=x\Rightarrow PB$ bisects $\angle CPD$.

(c)

(1) $T(k,x)-T(k,x+1)$

$$=\frac{k!}{x(x+1)\cdots(x+k)}-\frac{k!}{(x+1)(x+2)\cdots(x+k)(x+k+1)}$$

$$=\frac{k!\,(x+k+1)-k!\,x}{x(x+1)\cdots(x+k)(x+k+1)}$$

$$=\frac{(k+1)!}{x(x+1)\cdots(x+k+1)}=T(k+1,x).$$

7 (ii) Test $n = 1$:

$$\text{LHS} = T(1,x) = \frac{{}^1C_0}{x} - \frac{{}^1C_1}{x+1} = \frac{1}{x} - \frac{1}{x+1} = \frac{1}{x(x+1)} = \frac{1!}{x(x+1)} = \text{RHS}$$

So the statement is true for $n = 1$

Assume true for some integer $n = k$.

ie $T(k,x) = \dfrac{k!}{x(x+1)(x+2)\cdots(x+k)}$

We need to prove the statement is true for $n = k + 1$

ie $T(k+1,x) = \dfrac{(k+1)!}{x(x+1)(x+2)\cdots(x+k)(x+k+1)}$

$$\begin{aligned}\text{LHS} &= T(k+1,x)\\ &= T(k,x) - T(k,x+1) \qquad \boxed{\text{from (i)}}\\ &= \frac{(k+1)!}{x(x+1)(x+2)\cdots(x+k)(x+k+1)}\\ &= \text{RHS}\end{aligned}$$

(iii) Substitute $x = 1/2$ into both sides of the result from (ii) and simplify

$$\frac{{}^nC_0}{\frac{1}{2}} - \frac{{}^nC_1}{\frac{1}{2}+1} + \frac{{}^nC_2}{\frac{1}{2}+2} + \dots (-1)^n \frac{{}^nC_n}{\frac{1}{2}+n} = \frac{n!}{\frac{1}{2}\cdot\frac{3}{2}\cdot \cdots (n+\frac{1}{2})}$$

$$= \frac{2^{n+1}\, n!}{1\cdot 3\cdot 5\cdot (2n+1)}$$

$$\frac{{}^nC_0}{1} - \frac{{}^nC_1}{3} + \frac{{}^nC_2}{5} + \dots (-1)^n \frac{{}^nC_n}{2n+1} = \frac{2^n\, n!}{1\cdot 3\cdot 5 \cdots (2n+1)}$$

8.

(a) $V = \pi \int_0^{\pi} (c - \sin x)^2 \, dx$   2

$= \pi \int_0^{\pi} c^2 - 2c\sin x + \frac{1}{2} - \frac{1}{2}\cos 2x \, dx$

$= \pi \left[ c^2 x + 2c\cos x + \frac{1}{2}x - \frac{1}{4}\sin 2x \right]_0^{\pi}$

$= \pi \left[ c^2\pi - 2c + \frac{1}{2}\pi - 0 - (0 + 2c + 0 - 0) \right]$

$= \pi \left[ c^2\pi - 4c + \frac{1}{2}\pi \right]$   1

$V' = \pi(2c\pi - 4)$

$V'' = \pi(2\pi) > 0$   1

Min occurs when $V' = 0$   (5)

i.e. $2c\pi = 4$

$c = \frac{2}{\pi}$   1

(b) $f(\theta) = \frac{2 - \cos\theta}{\sin\theta}$, $0 < \theta < \frac{\pi}{2}$

$f'(\theta) = \frac{1 - 2\cos\theta}{\sin^2\theta}$ (by quotient rule)

Stat pt where $f'(\theta) = 0$

$1 - 2\cos\theta = 0$

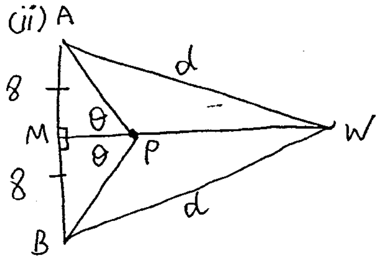
$\cos\theta = \frac{1}{2}$   $\theta = \frac{\pi}{3}$   3

when $\theta = 1$, $f'(\theta) > 0$   $\therefore$ Min value

$\theta = 1.2$, $f'(\theta) < 0$   when $\theta = \frac{\pi}{3}$

$f(\frac{\pi}{3}) = \frac{2 - \cos\frac{\pi}{3}}{\sin\frac{\pi}{3}} = \frac{2 - \frac{1}{2}}{\frac{\sqrt{3}}{2}} = \sqrt{3}$

b (ii)




1 Pythag

+ 1 cosec or cot

3 correct.

$PW = \sqrt{d^2-64} - 8\cot\theta$

$L = 2PA + PW$

$= 2\times 8\operatorname{cosec}\theta + \sqrt{d^2-64} - 8\cot\theta$

3

$= 16 \cdot \frac{1}{\sin\theta} - 8\frac{\cos\theta}{\sin\theta} + \sqrt{d^2-64}$

$= 8\left(\frac{2-\cos\theta}{\sin\theta}\right) + \sqrt{d^2-64} = 8f(\theta) + \sqrt{d^2-64}$

Clearly $\sin\theta \le 1$

$\sin\theta = \frac{8}{AP}$. Now $d = AW \ge AP$ (triangle inequality)

So $\sin\theta = \frac{8}{AP} \ge \frac{8}{d}$

ie $\frac{8}{d} \le \sin\theta \le 1$

iii) If $d = 20$ $\quad L = 8f(\theta) + \sqrt{336}$

$L' = 8f'(\theta)$

Min occurs when $f'(\theta) = 0$

2

by part (ii) min value of $L$ is

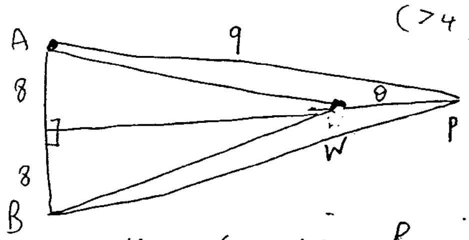
$L = 8\sqrt{3} + \sqrt{336}$.

when min, length of $MP$ is $8\cot\frac{\pi}{3} = \frac{8}{\sqrt{3}}$

We can use this value of $\theta$ because $\frac{8}{20} \le \sin\theta \le 1 \Rightarrow \sin^{-1}(2/5) \le \theta \le \pi/2$ and clearly $\theta = \pi/3$ satisfies this inequality.

(iv) Note $MW = \sqrt{81-64} = \sqrt{17} \not< \frac{8}{\sqrt{3}}$

$(>4) \qquad (<4)$




2

Key idea is $P$ is beyond $W$.

**More appropriate alternative solution:**
We can’t use the same method because with $d$ = 9 we get
$\frac{8}{9} \le \sin\theta \le 1 \Rightarrow \sin^{-1}(8/9) \le \theta \le \pi/2$ and clearly $\theta = \pi/3$ does NOT satisfy this inequality.
So in this range we have a 1:1 function for $L$ (you can quickly show that it is increasing) so all we need to is test the end points of $\sin^{-1}(8/9) \le \theta \le \pi/2$ ie substitute $\theta = \sin^{-1}(8/9)$ and $\theta = \pi/2$ into the formula for $L$ and take the minimum value of $L$ resulting.